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OPTIMAL SHAPE AND NON-HOMOGENEITY OF A NON-UNIFORMLY COMPRESSED COLUMN†

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Abstract—The best possible distribution of Young's modulus and/or the cross-sectional area is found for a column which, for a given volume and length, carries the maximum possible axial loads which are non-uniformly distributed along its length and concentrated at the end-points. The column is elastically clamped at one end and free at the other, where the concentrated axial load is applied. The design variables are subject to upper and lower bounds. Sufficient optimality conditions are derived for a given function to be a solution of the optimization problem. The procedure to determine the optimal solutions is described. Numerical results are obtained by employing an iterative computational technique.

1. INTRODUCTION

We consider the problem of maximizing the total axial load which is non-uniformly distributed along the length and concentrated at the end of a column of a given volume. This maximization can be achieved by optimally designing the distribution of the non-homogeneity and/or the cross-sectional area. We first solve the problem of optimizing a non-homogeneous column with respect to its shape and then the problem of optimizing a column with respect to its shape and Young's modulus with inequality constraints imposed on both the design variables. In the first problem we have minimum and maximum thickness constraints, while in addition to these, we have in the second problem upper and lower bounds on the Young's modulus. The column is elastically clamped at one end and free at the other, where an axial load is acting. We assume that the instability occurs at the fundamental mode of buckling. Solutions are found for columns with a variety of cross-sectional geometries.

In mathematical terms, the optimal design problems under investigation are equivalent to the problem of maximizing the lowest eigenvalue of a linear second-order ordinary differential equation with variable coefficients known as isoperimetric problems in the calculus of variations[1]. Certain coefficient functions (design variables) in the differential equation are to be varied subject to some integral conditions. In the last section we formulate our results in the form of an isoperimetric inequality which gives an upper bound on the lowest eigenvalue of the differential equation.

The present investigation differs from previous studies on optimal columns basically in considering a general form of loading, allowing for the longitudinal non-homogeneity of the material, and finally optimizing the column with respect to both shape and non-homogeneity.

With the exception of [2, 3], where the weight of the column is taken into account, only concentrated loads acting at the ends were examined in previous papers on the optimal design of conservatively loaded columns[4–18]. Furthermore, the non-homogeneity of the columns has been given attention only in [8, 9]. In [8], the numerical results are given only for homogeneous cases and in [9] the optimal distribution of non-homogeneity and shape is computed by first assuming the deflection curve in a special case. On the other hand, the optimal Young's modulus of structural elements of constant cross-sectional area has been determined in a few cases. Klosowicz and Lurie[19] determined the optimal non-homogeneity of a torsional bar and Rammerstorfer[20] that of a vibrating beam. In [20] no lower bound is imposed on the Young's modulus, and consequently there are points where this becomes zero. The same situation has been observed in the treatment of shape optimization problems without any thickness constraints[2, 4, 5, 7–9, 11–14]. As this is clearly undesirable from a practical point of view, either a minimum thickness or a maximum stress constraint has been imposed in some formulations in order to avoid this situation[3, 6, 10, 15–18]. In our case, the design variables,

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the Young's modulus and/or the cross-sectional area, are subject to upper and lower nds. Optimal columns with elastic clamping have been considered in [8, 12].

n Section 2 we derive the basic equations of the problem. Optimality conditions are ined in Section 3 by a method suggested by Barnes [16, 17] and a qualitative analysis of the nal forms is given.

'o obtain the numerical results in Section 4, we employ an iterative computational scheme ar to that used by Niordson [21]. In this section the relations between the critical buckling and the volume, the length, the cross-sectional geometry and the non-homogeneity of the nn are established by an isoperimetric inequality.

2. FORMULATION OF THE PROBLEM

'e consider an untwisted column of length L, volume V and cross-sectional area $A(X)$: X is the coordinate along the unbuckled state of the column (Fig. 1). The column is :ally clamped at $X = 0$ and carries a non-uniformly distributed axial load $0 \leq \lambda_0 Q(X) < \infty$ iit length along it. There may be an axial compressive load $\lambda_0 P$ at the free end ($X = L$). ll λ_0 the load factor. The column is made of an isotropic, linearly elastic material which ongitudinal non-homogeneity described by Young's modulus $E(X)$. Denoting by $I(X)$ the ıt of inertia of the cross section about an axis through the centroid of the column dicular to the plane of bending, and by $Y(X)$ the deflection function, the equation of rium is

$$[EIY'']'' + \lambda_0\left[\left(\int_X^L Q(t)\,\mathrm{d}t + P\right)Y'\right]' = 0 \tag{2.1}$$

to the boundary conditions

$$Y = 0, \quad Y' = \beta_0 EIY'' \quad \text{at} \quad X = 0, \tag{2.2}$$

$$EIY'' = 0, \quad (EIY'')' + \lambda_0 PY' = 0 \quad \text{at} \quad X = L \tag{2.3}$$

'prime denotes differentiation with respect to X, and β_0 characterizes the rotational of the support with $\beta_0 = 0$ for a rigidly clamped column, $\beta = \infty$ for an ideal hinge. We sider the optimal design problem of maximizing the load factor λ_0 by (i) determining -sectional area distribution $A(X)$, for *a priori* specified functions $E(X)$, $Q(X)$ and P, the volume and thickness constraints

$$\int_0^L A(X)\,\mathrm{d}X = V \tag{2.4}$$

$$0 \leq A_{min} \leq A(X) \leq A_{max} \quad \text{for} \quad 0 \leq X \leq L \tag{2.5}$$

w A_{min}, A_{max} are given constants; (ii) determining both $A(X)$ and the distribution of

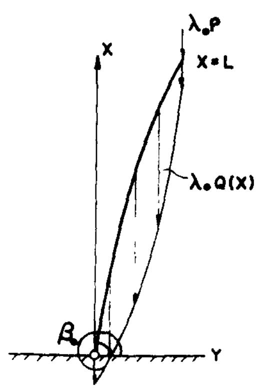

Fig. 1. Fundamental mode of buckling.

non-homogeneity $E(X)$, for *a priori* specified $Q(X)$ and P, subject to (2.4), (2.5) and

$$\int_0^L E(X)\,dX = S \tag{2.6}$$

$$0 \leq E_{min} \leq E(X) \leq E_{max} \quad \text{for} \quad 0 \leq X \leq L \tag{2.7}$$

where S, E_{min}, E_{max} are given constants, such that (2.1)–(2.3) will have a non-trivial solution $Y(X)$ with no internal nodes.

The condition on internal nodes is imposed to ensure that $Y(X)$ is the lowest mode of buckling. Clearly V and S should satisfy $LA_{min} < V < LA_{max}$ and $LE_{min} < S < LE_{max}$ to guarantee feasibility.

We introduce the following dimensionless quantities:

$$x = \frac{X}{L},\quad y(x) = \frac{Y(xL)}{L},\quad e(x) = \frac{L}{S}E(xL),\quad a(x) = \frac{L}{V}A(xL),$$

$$i(x) = \frac{I(xL)}{VL},\quad p = P\Big/\int_0^L Q(X)\,dX,\quad q_0(x) = LQ(xL)\Big/\int_0^L Q(X)\,dX, \tag{2.8}$$

$$q(x) = \int_x^1 q_0(\xi)\,d\xi.$$

We assume that there exists a relation between $i(x)$ and $a(x)$, expressible in the form

$$i(x) = k_n a^n(x) \tag{2.9}$$

where k_n is a dimensionless constant depending on n and the cross-sectional geometry, and $n = 1, 2$ or 3. For sandwich columns of rectangular cross section with fixed width and variable face-sheet thickness, $n = 1$ and $k_1 = H^2/L^2$, where $2H$ = fixed core thickness. Solid columns with geometrically similar cross-sections have $n = 2$ and $k_2 = \bar{k}V/L^3$, where $\bar{k}$ depends on the cross-sectional shape. For solid columns of rectangular cross section of fixed width and variable depth, $n = 3$ and $k_3 = V^2/12B^2L^4$, where B = width. We refer to [10] for further details. In the case of thin-walled circular columns of similar cross sections, $n = 1$, $k_1 = D^2/8L^2$ if the design variable is the wall thickness [3], and $n = 3$, $k_3 = V^2/8\pi^2L^4t^2$ if the design variable is the diameter; where D = cross-sectional diameter and t = wall thickness [22]. We define

$$\lambda = \frac{L^2\lambda_0}{SVk_n}\int_0^L Q(X)\,dX,\quad \beta = \frac{\beta_0 SVk_n}{L}. \tag{2.10}$$

Substituting (2.8), (2.9) and (2.10) into (2.1), we have

$$[ea^n y'']'' + \lambda[(q(x)+p)y']' = 0. \tag{2.11}$$

We expect the optimal $a(x)$ and $e(x)$ functions to be continuous but not continuously differentiable in view of the inequality constraints (2.5) and (2.7). Therefore, by setting $M = ea^n y''$, the differential eqn (2.11) is transformed into a form which does not involve the derivatives of $a(x)$ and $e(x)$. Various advantages of a formulation in terms of the bending moment M have already been noted by Masur [18]. After some transformations and using the boundary condition $(ea^n y'')' + \lambda p y' = 0$ at $x = 1$, we obtain

$$\left(\frac{M'}{q(x)+p}\right)' + \lambda\frac{M}{ea^n} = 0 \tag{2.12}$$

subject to

$$M'(0) + \lambda\beta(1+p)M(0) = 0,\quad M(1) = 0. \tag{2.13}$$

Let $\mathscr{PC}$ denote the class of functions $f(\cdot)$ that are piecewise continuously differentiable in the sense that they are continuous everywhere on $[0, 1]$ and continuously differentiable there, with the possible exception of at most a finite number of points where the derivative of $f(\cdot)$ shall have well-defined limiting values both from the left and the right.

Definition. $\mathscr{A}$ denotes the class of functions $a \in \mathscr{PC}$ satisfying

$$\int_0^1 a(x)\,\mathrm{d}x = 1, \quad 0 \le a_{\min} \le a(x) \le a_{\max} \quad \text{for} \quad x \in [0, 1]. \tag{2.14}$$

$a \in \mathscr{A}$ is called an admissible a-function.

Definition. ξ denotes the class of functions $e \in \mathscr{PC}$ satisfying

$$\int_0^1 e(x)\,\mathrm{d}x = 1, \quad 0 \le e_{\min} \le e(x) \le e_{\max} \quad \text{for} \quad x \in [0, 1]. \tag{2.15}$$

$e \in \xi$ is called an admissible e-function.

We note that (2.14), (2.15) correspond to the non-dimensional forms of (2.4)–(2.7). Furthermore, the inequalities $a_{\min} < 1 < a_{\max}$ and $e_{\min} < 1 < e_{\max}$ should be satisfied to guarantee feasibility.

Definition. $\mathscr{M}$ denotes the class of twice continuously differentiable functions $M(\cdot)$ on $[0, 1]$ satisfying (2.13). $M \in \mathscr{M}$ is called an admissible M-function.

We now state the objectives of the paper in the form of extremal eigenvalue problems.

Problem I. Determine the optimal shape $a \in \mathscr{A}$ for given e, q, p such that the lowest eigenvalue λ of (2.12), (2.13) is as large as possible.

Problem II. Determine the optimal distributions of the shape $a \in \mathscr{A}$ and the non-homogeneity $e \in \xi$ for given q, p such that the lowest eigenvalue λ of (2.12), (2.13) is as large as possible.

We note that the problem of finding the optimal e-function, $e \in \xi$, for a given shape a, is equivalent to Problem I with $n = 1$.

3. OPTIMALITY CONDITIONS

We derive the optimality conditions for Problems I and II by making use of a theorem of Hestenes[23]. Consider the problem of minimizing

$$J_0(u) = \int_0^1 F_0(t, u(t))\,\mathrm{d}t \tag{3.1}$$

on $u \in \mathscr{PC}$ satisfying the constraints

$$\int_0^1 F_i(t, u(t))\,\mathrm{d}t = D_i \quad (i = 1, 2, \dots, l), \quad u_1 \le u \le u_2 \tag{3.2}$$

where u_1, u_2, D_i are fixed constants and F_i, $i = 0, 1, \dots, l$, are given continuous functions on $[0, 1] \times [u_1, u_2]$.

Noting that the inequality constraints in (3.2) define an admissible class R_0 in the sense of Hestenes (p. 203[23]), we have (Th. 5.1, p. 215[23]) the following theorem.

Theorem 3.1. Suppose that $u_0 \in \mathscr{PC}$, satisfying (3.2), minimizes J_0. Then there exist multipliers $\eta_0 \ge 0, \eta_1, \dots, \eta_l$, not all zero, such that

$$\sum_{i=0}^{l} \eta_i F_i(t, u) \ge \sum_{i=0}^{l} \eta_i F_i(t, u_0(t)), \quad 0 \le t \le 1 \tag{3.3}$$

holds for all admissible elements (t, u). Conversely, if there exist multipliers $\eta_0 > 0, \eta_1, \ldots, \eta_l$ such that (3.3) holds, then u_0 minimizes J_0 in the class of admissible functions u satisfying (3.2).

Problems I and II are in a form to which Theorem 3.1 cannot be directly applied. Thus we next derive a sufficient condition for a function $a_0 \in \mathscr{A}$ to be a solution of Problem I.

Theorem 3.2. Let $M_0 \in \mathscr{M}$ be an eigenfunction of (2.12) for a given e with some $a = a_0 \in \mathscr{A}$, corresponding to the lowest eigenvalue $\lambda(a_0)$ of the problem. Then a_0 is a solution of Problem I if

$$\int_0^1 \frac{M_0^2}{e a_0^n} \mathrm{d}x \leq \int_0^1 \frac{M_0^2}{e a^n} \mathrm{d}x \tag{3.4}$$

for every $a \in A$.

Proof. The Rayleigh quotient associated with (2.12), (2.13) is

$$R(a, M) = \frac{\int_0^1 (q(x)+p)^{-1} M'^2 \, \mathrm{d}x}{\beta M^2(0) + \int_0^1 \frac{M^2}{e a^n} \mathrm{d}x}, \quad a \in \mathscr{A}, \quad M \in \mathscr{M}. \tag{3.5}$$

From Raleigh's principle[10], it follows that

$$\lambda(a) = \min_{M \in \mathscr{M}} R(a, M) = R(a, M_0) \leq R(a_0, M_0) = \lambda(a_0). \tag{3.6}$$

The inequality in (3.6) follows from (3.4). Relation (3.6) shows that $a_0 \in \mathscr{A}$ is a solution of Problem I. We note that in obtaining (3.5), we provisionally assume $M'(1)M(1)(q(1)+p)^{-1} = 0$ for $p \geq 0$. This is justified in Section 4.

Theorem 3.2 permits us to construct a solution $a_0 \in \mathscr{A}$ satisfying (3.4). Once this has been done, we can check that this solution is optimal by using the sufficiency part of Theorem 3.1. We first reformulate Problem I in the light of Theorem 3.2.

Problem Ia. Determine $a_0 \in \mathscr{A}$, $M_0 \in \mathscr{M}$ such that

$$\min_{a \in \mathscr{A}} \int_0^1 \frac{M_0^2}{e a^n} \mathrm{d}x = \int_0^1 \frac{M_0^2}{e a_0^n} \mathrm{d}x, \tag{3.7}$$

where $M_0 \in \mathscr{M}$ is an eigenfunction corresponding to the lowest eigenvalue of (2.12) with $a = a_0$.

Although M_0 is an unknown function in (3.7), this causes no difficulty when applying Theorem 3.1 and it is determined together with $a = a_0$.

Theorem 3.3. The maximum of $R(a, M_0)$, defined in (3.5), is attained at $a = a_0 \in \mathscr{A}$ if there exist constants $\eta_0 > 0, \eta$ such that

$$\min_{a \in \mathscr{A}} F(a, M_0) \equiv \min_{a \in \mathscr{A}} \left[\eta_0 \frac{M_0^2}{e a^n} + \eta a \right] = F(a_0, M_0). \tag{3.8}$$

where M_0 is as defined in Problem Ia.

Proof. In (3.1), $F_0(t, a) = M_0^2/ea^n$ in view of the formulation of Problem I given in Problem Ia. Clearly $F_1(t, a) = a$ since $a \in \mathscr{A}$. Application of the sufficiency part of Theorem 3.1 with F_0 and F_1 yields the conclusion (3.8).

We next derive the optimality condition for Problem I.

Theorem 3.4. The function $a_0 \in \mathscr{A}$ which maximizes the lowest eigenvalue of (2.12), (2.13) satisfies

$$a_0 = \begin{cases} a_{\min} & \text{if} \quad M_0^{2/(n+1)} < a_{\min}(e\eta)^{1/(n+1)} \\ (M_0^2/e\eta)^{1/(n+1)} & \text{if} \quad a_{\min}(e\eta)^{1/(n+1)} \leq M_0^{2/(n+1)} < a_{\max}(e\eta)^{1/(n+1)} \\ a_{\max} & \text{if} \quad M_0 2/(n+1) \geq a_{\max}(e\eta)^{1/(n+1)} \end{cases} \tag{3.9}$$

where η is a positive constant and $M_0 \in \mathcal{M}$ is a solution of (2.12) with $a = a_0$.

Proof. The optimal solution a_0 is constructed by so defining it that (3.8) is satisfied for each $x \in [0, 1]$. The function $F(a, M_0)$, defined in (3.8), is convex in a for every $x \in [0, 1]$. Therefore $F(a, M_0)$ will be minimized at $a = a_0 \in \mathcal{A}$ if

$$\begin{aligned} a_0 &= a_{\min} && \text{when} \quad \frac{\partial F(a_{\min})}{\partial a} \geq 0 \\ a_0 &= \left(\frac{n\eta_0 M_0^2}{e\eta}\right)^{1/(n+1)} && \text{when} \quad \frac{\partial F(a_{\min})}{\partial a} < 0 \quad \text{and} \quad \frac{\partial F(a_{\max})}{\partial a} > 0 \\ a_0 &= a_{\max} && \text{when} \quad \frac{\partial F(a_{\max})}{\partial a} \leq 0 \end{aligned} \tag{3.10}$$

where $\partial F(a_{\min})/\partial a = \partial F/\partial a$ evaluated at $a = a_{\min}$. The expression for a_0 in the second line of (3.10) follows from the condition that $\partial F/\partial a = 0$ if $a_{\min} < a_0 < a_{\max}$. By inserting (3.8) into (3.10) and setting $\eta_0 = 1/n$, we deduce (3.9). It can be seen that η is positive by noting that the results are untenable when a negative η is inserted in (3.9). In fact, from (2.14) and (3.9) it follows that $0 < \eta < \max\limits_{0 \leq x \leq 1} (M_0^2/e)$.

The relation $(3.9)_2$ was derived by various methods[5, 14, 18] for the unconstrained optimal columns. The physical interpretation of this condition was first given by Masur[24].

We note that the bending stress σ of a thin-walled column is proportional to Mai^{-1} when $n = 3$[22]. Hence $\sigma\alpha Ma^{-2}$ by (2.9). From (3.9), it follows that

$$\sigma\alpha e^{1/2} \quad \text{for} \quad a_{\min} < a_0 < a_{\max}. \tag{3.11}$$

In the case of a homogeneous column ($e \equiv 1$), we have $\sigma = \text{constant}$ for the optimal column. This result was conjectured by Feigen[22] in 1952 for thin-walled columns with no minimum or maximum thickness constraint imposed on them. This conjecture was proved rigorously for columns carrying an axial point load by Tadjbakhsh and Keller[5] in 1962.

We now describe the procedure for applying (3.9) in computing the optimal solution $a_0 \in \mathcal{A}$ of Problem I. In the next section we give a computational technique for obtaining numerical results based on this solution procedure.

We shall trace the optimal path starting from the point $x = 1$, the free end of the column, and moving backwards to $x = 0$, the elastically clamped end of the column.

Since $M(1) = 0$, it follows that in a left neighbourhood of the end point $x = 1$, i.e. for $x \in (x_1, 1]$ where x_1 is an unknown constant, we have $M_0^{2/(n+1)} < a_{\min}(e\eta)^{1/(n+1)}$ for $a_{\min} > 0$. From (3.9) it follows that $a_0(x) = a_{\min}$ for $x \in [x_1, 1]$. By inserting this value of a_0 into (2.12), we obtain

$$\left(\frac{M_0'}{q(x)+p}\right)' + \lambda \frac{M_0}{e a_{\min}^n} = 0, \quad x_1 \leq x < 1. \tag{3.12}$$

$a_0(x) \equiv a_{\min}$ cannot be a solution of the problem owing to the assumption $a_{\min} < 1$ and (2.14). Therefore there exists a greatest x_1 in (0, 1) such that $M_0^{2/(n+1)}(x_1) = a_{\min}(e\eta)^{1/(n+1)}$. Since $a_{\min} < a_{\max}$ and M_0 is a continuous function, we have $a_{\min}(e\eta)^{1/(n+1)} \leq M_0^{2/(n+1)} < a_{\max}(e\eta)^{1/(n+1)}$ for some non-zero interval $[x_2, x_1]$ where $0 \leq x_2 < x_1$ is an unknown constant. Hence we have

$$a_0 = \frac{M_0^{2/(n+1)}}{(e\eta)^{1/(n+1)}}, \quad x_2 \leq x < x_1. \tag{3.13}$$

Inserting (3.13) into (2.12), we obtain

$$\left(\frac{M_0'}{q(x)+p}\right)' + \lambda\eta^{n/(n+1)}\frac{M_0^{(1-n)/(n+1)}}{e^{1/(n+1)}} = 0, \quad x_2 \leq x < x_1. \tag{3.14}$$

After this point, two cases have to be distinguished with regard to the number N of intervals, where $a_0 = a_{\min}$, $a_0 = a_{\max}$ or $a_{\min} < a_0 < a_{\max}$, depending on whether $e(x)$ is a non-decreasing function or merely some piecewise analytic function. When $e(x)$ is non-decreasing the case is less complicated and this we shall treat first.

Let $e(x)$ be a non-decreasing function. Since eigenfunctions are unique only up to a scalar multiple, we may assume that M_0 has been scaled, so that

$$M_0(0) = 1. \tag{3.15}$$

Thus $M_0(x)$ is a decreasing function on $[0, 1]$, since it satisfies the Sturm–Liouville system (2.12), (2.13) and (3.15)[25]. If $a_{\max}$ is sufficiently lárge, we shall have $M_0^{2/(n+1)} < a_{\max}(e\eta)^{1/(n+1)}$ for all $0 \leq x \leq x_1$. This amounts to the assumption that the optimal column nowhere achieves the maximum allowable thickness $a_{\max}$, and consequently $N = 2$. In this case, the solution $M_0 \in \mathcal{M}$ of (3.12) and (3.14) yields the optimal shape when substituted into (3.9). Otherwise, we have one additional interval $[0, x_2]$ where $a_0 = a_{\max}$, so that $N = 3$. Since M_0 is a monotonic function and $e(x)$ is non-decreasing, $a_0(x)$ is a non-increasing function due to (3.9). Consequently the possibility of any more intervals is excluded. We first solve the problem for $N = 2$ with $x_2 = 0$ and check whether $a_0(0) \leq a_{\max}$ is violated. If $a_0(0) > a_{\max}$, a new interval $[0, x_2]$ is added, where M_0 is computed from (3.12) with $a_{\min}$ replaced by $a_{\max}$.

When $e(x)$ is an arbitrary piecewise analytic function, the optimal shape function $a_0(x)$ will in general have various intervals where $a_0 = a_{\min}$ or $a_0 = a_{\max}$, the intervals being connected by appropriate arcs. The first two intervals will be those described by (3.12)–(3.14), and in the subsequent intervals (3.12) with $a_0 = a_{\min}$ or $a_{\max}$ or (3.14) will again apply. It is not possible to determine the exact number N of intervals *a priori*, since N depends on the unknown constant η which, in turn, depends on $a_{\min}$, $a_{\max}$, β, $q(x)$, p and $e(x)$. For this reason, the solution procedure is basically a trial-and-error technique.

We solve the problem by first assuming $N = 2$ and increasing N by one whenever the relevant inequalities in (3.9) are not satisfied. Thus $a_{\min} \leq a_0 \leq _{\max}$ as well as $M_0^{2/(n+1)} \lesseqgtr a_{\min}$ (or $a_{\max}) \cdot (e\eta)^{1/(n+1)}$ should be checked for every N. In this way, the solution proceeds toward the point $x = 0$ and will reach it after a finite number of trials.

At the junction points we have the continuity relations

$$\begin{gathered} a_{0L}(x_i) = a_{0R}(x_i) \\ M_{0L}(x_i) = M_{0R}(x_i), \quad M_{0L}'(x_i) = M_{0R}'(x_i), \quad i = 1, 2, \ldots, N-1 \end{gathered} \tag{3.16}$$

where the subscripts L and R denote the quantities to the left and right of x_i respectively. Equations (2.13), (2.14), (3.15), (3.16) provide $3N + 1$ equations for the unknown constants. As unknowns we have $2N$ integration constants, $N - 1$ interval lengths x_i, a Lagrange multiplier η and the eigenvalue λ, i.e. $3N + 1$ unknowns in all. Thus the formulation poses a well-determined problem for the optimal column.

We next derive the optimality condition for Problem II. Now $e \in \xi$ and $a \in \mathcal{A}$ are design variables. Theorem 3.2 with obvious modifications applies to this case and permits us to reformulate Problem II as follows.

Problem IIa. Determine $a_0 \in \mathcal{A}$, $e_0 \in \xi$, $M_0 \in \mathcal{M}$ such that

$$\min_{\substack{a \in \mathcal{A} \\ e \in \xi}} \int_0^1 \frac{M_0^2}{ea^n}\,\mathrm{d}x = \int_0^1 \frac{M_0^2}{e_0 a_0^n}\,\mathrm{d}x, \tag{3.17}$$

and $M_0 \in \mathcal{M}$ is an eigenfunction corresponding to the lowest eigenvalue of (2.12) with $a = a_0$, $e = e_0$. Here a_0, e_0 correspond to the optimal solutions for Problem II.

We denote the Rayleigh quotient associated with Problem II by $R(a, e, M)$, which is given in (3.5).

Theorem 3.5. The maximum of $R(a, e, M_0)$, defined in (3.5), is attained at $a = a_0 \in \mathcal{A}$, $e = e_0 \in \xi$, if there exist constants $\eta_0 > 0$, η_1, η_2 such that

$$\min_{\substack{a\in\mathcal{A}\\ e\in\xi}} \left[\eta_0 \frac{M_0}{ea^n} + \eta_1 a + \eta_2 e \right] = \eta_0 \frac{M_0}{e_0 a_0^n} + \eta_1 a_0 + \eta_2 e_0 \tag{3.18}$$

where M_0 is as defined in Problem IIa.

Proof. Except for minor modifications, the proof is the same as that of Theorem 3.3.

The optimality condition for Problem II is given in the next theorem.

Theorem 3.6. The functions $a_0 \in \mathcal{A}$, $e_0 \in \xi$ which maximize the lowest eigenvalue of (2.12), (2.13) satisfy

$$a_0 = \begin{cases} a_{\min} & \text{if } M_0^{2/(n+2)} < a_{\min}(\eta_1^2/\eta_2)^{1/(n+2)} \\ (\eta_2 M_0^2/\eta_1^2)^{1/(n+2)} & \text{if } a_{\min}(\eta_1^2/\eta_2)^{1/(n+2)} \leq M_0^{2/(n+2)} < a_{\max}(\eta_1^2/\eta_2)^{1/(n+2)} \\ a_{\max} & \text{if } M_0^{2/(n+2)} \geq a_{\max}(\eta_1^2/\eta_2)^{1/(n+2)} \end{cases}$$

$$e_0 = \begin{cases} e_{\min} & \text{if } M_0^{2/(n+2)} \leq e_{\min}(\eta_2^{n+1}/\eta_1^n)^{1/(n+2)} \\ (\eta_1^n M_0^2/\eta_2^{n+1})^{1/(n+2)} & \text{if } \leq e_{\min}(\eta_2^{n+1}/\eta_1^n)^{1/(n+2)} \leq M_0^{2/(n+2)} < e_{\max}(\eta_2^{n+1}/\eta_1^n)^{1/(n+2)} \\ e_{\max} & \text{if } M_0^{2/(n+2)} \leq e_{\max}(\eta_2^{n+1}/\eta_1^n)^{1/(n+2)} \end{cases} \tag{3.19}$$

where η_1, η_2 are positive constants and $M_0 \in \mathcal{M}$ is a solution of (2.12) with $a = a_0$, $e = e_0$.

Proof. Except for minor modifications, the proof is the same as that of Theorem 3.4.

The optimal distribution of the shape and the non-homogeneity can be determined by tracing the column starting from $x = 1$ and moving backwards towards $x = 0$. In this case optimal a- and e-functions are decreasing and consequently $N \leq 3$. The essential character of the analysis is the same as that of Problem I given in the previous paragraphs.

4. NUMERICAL RESULTS AND DISCUSSION

For solving Problem I numerically, the optimality condition (3.9) suggests an iterative computational technique similar to that given in [21]. For this purpose, we need to study the behaviour of solutions near $x = 1$. As a consequence of (2.8), $q(x) \sim O[(1-x)^m]$, $m > 0$ near $x = 1$. We shall seek M_0 near $x = 1$ in the form

$$M_0(x) = b(1-x)^c + \cdots, \quad c > 0 \tag{4.1}$$

where b and c are constants to be determined and $c > 0$ is implied by (2.13). Inserting (4.1) into (3.12) and (3.14) and equating the coefficient of the leading term to zero, we find

$$c = 1 \quad \text{for} \quad p > 0, \quad c = 1 + m \quad \text{for} \quad p = 0. \tag{4.2}$$

The assumption made in obtaining (3.5) viz that $M'M(q(x)+p)^{-1}$ vanishes at $x = 1$, can now be verified. From (4.1), we compute $M'M(q(x)+p)^{-1} = b^2c(1-x)^{2c-1}[(1-x)^m + p] + \cdots$ which vanishes at $x = 1$ for both $p > 0$ and $p = 0$ owing to (4.2).

The behaviour of $M(x)$ near $x = 1$ leads us to define a bounded function $f(x)$ by the relation

$$f(x) = M_0'(x)(q(x)+p)^{-1}, \quad f(0) = -\lambda\beta \tag{4.3}$$

where the initial condition on $f(x)$ follows from (2.13) and (3.15). From (2.14) and (3.9), we deduce

$$\eta = \left[\left(\int_{S_u} M_0^{2/(n+1)} e^{-1/(n+1)}\,dx\right) \cdot \left(1 - a_{\min}\int_{S_{\min}} dx - a_{\max}\int_{S_{\max}} dx\right)^{-1}\right]^{n+1} \equiv H(M_0) \tag{4.4}$$

where S_u, $S_{\min}$ and $S_{\max}$ denote the subintervals over which a_0 satisfies $a_{\min} < a_0 < a_{\max}$, $a_0 = a_{\min}$ or $a_0 = a_{\max}$, respectively. Clearly $S_u \cup S_{\min} \cup S_{\max} = [0, 1]$. In particular, when $e(x)$ is a non-decreasing function, $S_u = [x_2, x_1]$, $S_{\min} = [x_1, 1]$ and $S_{\max} = [0, x_2]$. Furthermore, (3.5), (3.6), (3.15), (4.3) yield

$$\lambda_a = \frac{\int_0^1 (q+p)f^2\,dx}{\beta + \int_{S_{\max}} \frac{M_0^2}{e a_{\max}^n}\,dx + \int_{S_{\min}} \frac{M_0^2}{e a_{\min}^n}\,dx + \int_{S_u} \left(\frac{M_0^2}{\eta^{-n} e}\right)^{1/n+1} dx} \equiv I(M_0, f, \eta). \tag{4.5}$$

Formally integrating (3.12), (3.14) and using (4.3), we obtain

$$f(x) = \begin{cases} -\lambda_a a_{\min}^{-n} \int_{S_{\min}} \frac{M_0}{e}\,dx - \lambda_a\beta \equiv J_1(M_0, \lambda_a) & \text{if} \quad x \in S_{\min} \\ -\lambda_a \eta^{n/(n+1)} \int_{S_u} \frac{M_0^{(1-n)/(n+1)}}{e^{1/(n+1)}}\,dx - \lambda_a\beta \equiv J_2(M_0, \lambda_a, \eta) & \text{if} \quad x \in S_u \\ -\lambda_a a_{\max}^{-n} \int_{S_{\max}} \frac{M_0}{e}\,dx - \lambda_a\beta \equiv J_3(M_0, \lambda_a) & \text{if} \quad x \in S_{\max}. \end{cases} \tag{4.6}$$

An iterative scheme is defined as follows.

(i) Choose $x \in [0, 1]$, $f^{(0)}(x)$ and $\eta^{(0)}$ arbitrarily.
(ii) $M_0^{(i)}(x) = -\int_x^1 (q(t)+p) f^{(i)}(t)\,dt$. (4.7)
(iii) Normalize $M_o^{(i)}(x)$ so that (3.15) is satisfied.
(iv) Let $\bar{a}^{(i)}(x) = [M_o^{(i)2}(x)/\eta^{(i)} e(x)]^{1/(n+1)}$ and determine $S_{\min} = \{x | \bar{a}^{(i)}(x) \leq a_{\min}\}$, $S_{\max} = \{x | \bar{a}^{(i)}(x) \geq a_{\max}\}$ and $S_u = [0, 1] - (S_{\min} \cup S_{\max})$.
(v) $\eta^{(i+1)} = H(M_0^{(i)})$.
(vi) $\lambda^{(i+1)} = I(M_0^{(i)}, f^{(i)}, \eta^{(i)})$.
(vii) $f^{(i+1)}(x) = J_k(M_0^{(i)}, \lambda^{(i+1)}, \eta^{(i+1)})$, $k = 1, 2$ or 3.
(viii) If $f(x)$ and λ are non-stationary, go to (ii), else terminate.

The computational procedure was performed by introducing $j+1$ equally spaced points in the interval $0 \leq x \leq 1$ and defining the iterates $M_0^{(i)}$ at these points. The sequence of iterates converged rapidly and the numerical stability of the solutions was checked by computations based on different numbers of divisions of the interval.

We examine the effect of non-homogeneity on the optimal design by considering e-functions given by

$$e_0(x) = 1.0 \quad \text{(homogeneous column)}$$
$$e_1(x) = 1.2x + 0{,}4$$
$$e_2(x) = -1.2x + 1.6$$

which are characterized by the condition $\int_0^1 e(x)\,dx = 1$. Most of the numerical results are given for the loading $q_0(x) \equiv 1$, so that $q(x) = 1 - x$. This case corresponds to a uniformly distributed compressive load of magnitude 1 along the column. In the rest of the paper, the maximum eigenvalue of the fundamental buckling mode is denoted by λ_a in Problem I and by λ_{ac} in Problem II.

Figure 2 shows the ratio of maximum buckling load λ_a to the buckling load λ_{u} of a uniform column with the same volume and Young's modulus e, plotted against $a_{\min}$ with $p = \beta = 0$. Figure 3 gives the values of λ_a plotted against $a_{\min}$. Both figures are presented for e_i, $i = 0, 1, 2$ and $n = 1, 2, 3$ under the load $q_0 = 1$. With reference to these figures, the following observations can be made.

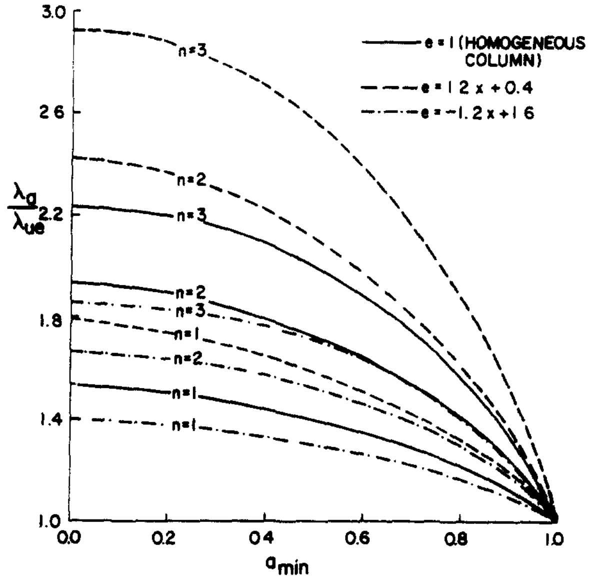

Fig. 2. Curves of λ_a/λ_{ue} plotted vs a_{min} for $n = 1, 2, 3$ and various functions $e(x)$ with $q_0 = 1$, $p = \beta = 0$.

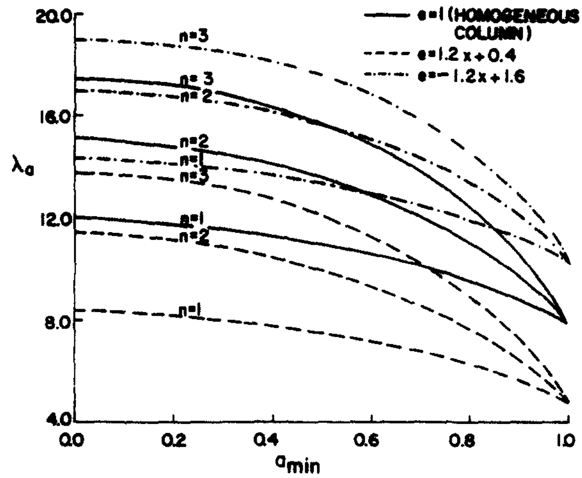

Fig. 3. Curves of λ_a plotted vs a_{min} for $n = 1, 2, 3$ and various functions $e(x)$ with $q_0 = 1$, $p = \beta = 0$.

(1) An increasing function $e(x)$ of x, e.g. $e = e_1(x)$, yields a higher $\lambda_r = \lambda_a/\lambda_{ue}$ in comparison with a decreasing function $e(x)$ of x, e.g. $e = e_2(x)$ (Fig. 2). But the value of λ_a is higher for $e = e_2(x)$ than for $e = e_1(x)$ (Fig. 3). Hence the efficiency of the optimal design is higher for increasing e-functions whereas the buckling load is higher for decreasing e-functions.

(2) The flatness of the curves in the vicinity of $a_{min} = 0$ implies that a relatively small thickness constraint does not appreciably reduce the optimal buckling loads compared with their unconstrained values.

(3) For higher values of n, the efficiency of the design increases.

Fig. 4 shows the optimal shape functions $a_0(x)$ for $n = 1$, 3 and $e = e_0$, e_2 with $q_0 = 1$, $p = \beta = 0$. We observe the following.

(1) The optimal shapes have a reverse taper at the clamped end for $e = e_2(x)$.

(2) From (3.9) and (4.2) we compute that $a(x)\alpha(1-x)^{4/(n+1)}$ near $x = 1$, since $m = 1$ for $q_0 = 1$. Thus $a(x)\alpha(1-x)^2$ for $n = 1$ and $a(x)\alpha(1-x)$ for $n = 3$. This explains the behaviour of the optimal shape near $x = 1$ in Fig. 4(a).

(3) The constraint $a_{min} = 0.2$ in Fig. 4(b) becomes effective at different lengths for each shape. This observation is again related to the above-mentioned behaviour of $a(x)$ near $x = 1$.

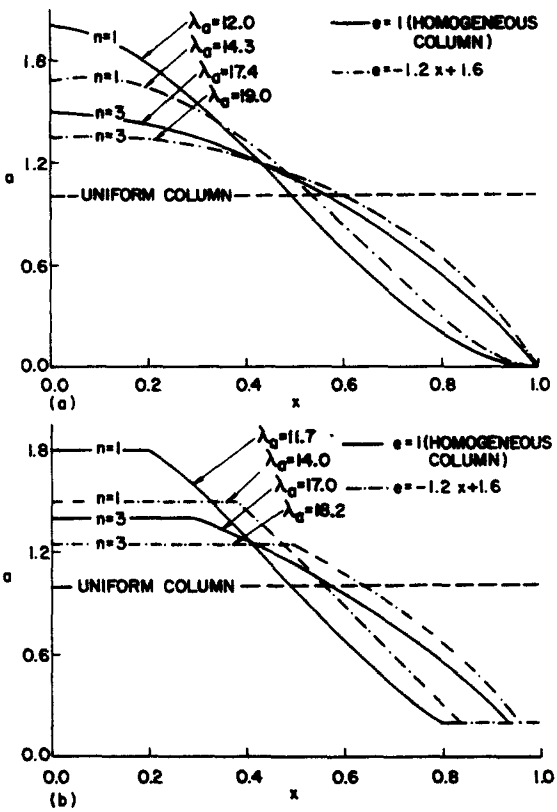

Fig. 4. Unconstrained and constrained optimal shapes with $q_0 = 1$, $p = \beta = 0$.

Figure 5 shows the effect of β on λ_a/λ_{ue} for the loadings $q_0 = 1$ and $q_0(x) = 2(1-x)$ with $n = 2$, $p = 0$. We observe that the ratio λ_a/λ_{ue} decreases rapidly with increasing β but tapers off afterwards. This behaviour is more pronounced for $q_0(x) = 2(1-x)$ than for $q_0 = 1$. We have $\lambda_a/\lambda_{ue} \to 1$ as $\beta \to \infty$.

Figure 6 gives the curves of λ_a/λ_u plotted vs p for e_0, e_1 and e_2 with $n = 2, 3$, $\beta = 0$, λ_u denoting the buckling load on a homogeneous uniform column. When the axial load p is infinite compared with the distributed load $q_0(x)$, λ_a/λ_u converges to well-defined limits. For the homogeneous columns ($e = 1$) these limits are known[10, 5]. In this case, $\lambda_a/\lambda_u \to 1.22$, 1.33 and 1.41 as $p \to \infty$ for $n = 1$, 2 and 3 respectively. The case $n = 2$ can be found in [5]; $n = 1$, 2 and 3 in [10].

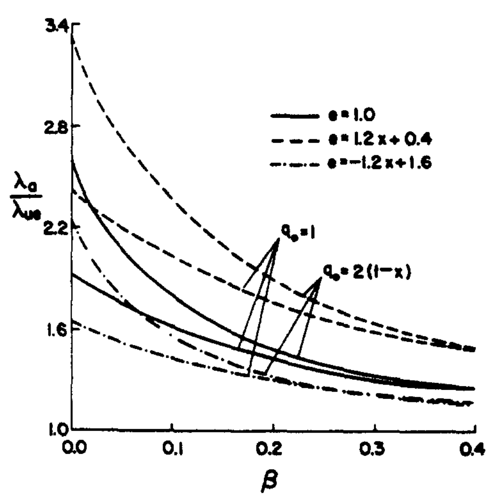

Fig. 5. Curves of λ_a/λ_{ue} plotted vs β for $n = 2$, $p = 0$. $a_{min} = 0$, $q_0 = 1$ and $q_0 = 2(1-x)$.

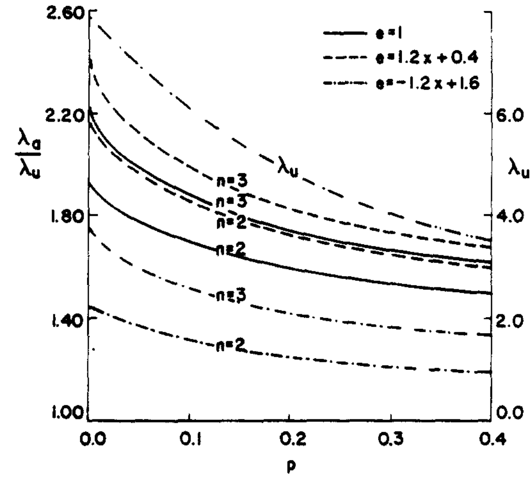

Fig. 6. Curves of λ_a/λ_u and λ_u plotted vs p for $\beta = 0$, $a_{min} = 0$ and $q_0 = 1$.

Figure 7 shows the optimal shapes $a(x)$ for various e, β and p for $n = 2$, $q_0 = 1$. Although the effects of β and p on the buckling load are similar as judged from Figs. 5 and 6, their effect on the optimal design differs: optimal shape tends to become more non-uniform ($a(0)$ increasing) with increasing β, but more uniform ($a(0)$ decreasing) with increasing p.

Figure 8 gives a comparative view of optimal designs at different loadings for $e = 1$, $n = 2$, $\beta = p = 0$. The curves show λ_a for $q_0(x) = 2(1-x)$, $0.5\pi \sin \pi x$, 1 and $2x$ plotted vs a_{min} in the interval $0 \leq a_{min} \leq 1$, where the end-point values correspond to an unconstrained and uniform column respectively. We observe that as the distributed load becomes more concentrated toward the clamped end of the column, i.e. $q_0(x) = 2(1-x)$, the buckling load of the optimal design increases.

To solve Problem II numerically, we employ a double iteration scheme which makes use of the computational procedure already formulated for Problem I in the previous paragraphs. Thereby, we avoid developing a new procedure based on (3.19).

We observe that for the unconstrained version of Problem II, i.e. $a_{min} = e_{min} = 0$ and $a_{max} = e_{max} = \infty$, the relation

$$e(x) = a(x), \quad 0 \leq x \leq 1$$

follows from (2.14), (2.15) and (3.19). Thus by simply replacing n by $n+1$ in the iterative scheme for Problem I, we can solve this case numerically. Let $e_0 = e_{min}$ for $x \in [x_3, 1]$, $e_0 = e_{max}$ for $x \in [0, x_4]$ where $0 \leq x_4 < x_3 \leq 1$. Since e_0 is a decreasing function and $N \leq 3$, this case

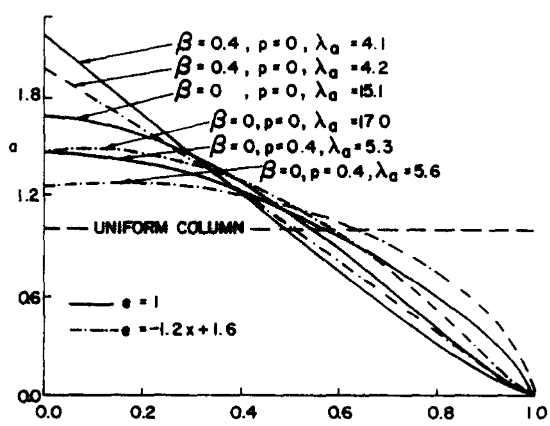

Fig. 7. Optimal shapes $a(x)$ for various values of β, p and $e(x)$ with $n = 2$, $q_0 = 1$.

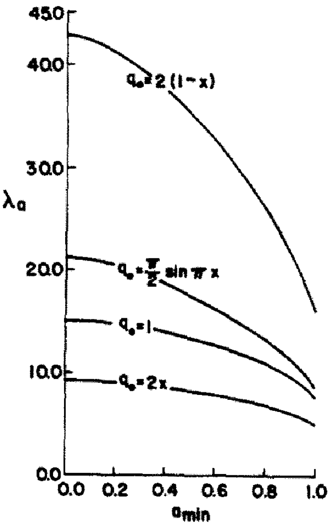

Fig. 8. Curves of λ_a plotted vs a_{min} for various loadings with $e = 1$, $n = 2$, $\beta = p = 0$.

corresponds to the general one. From (2.15) and (3.19), we obtain

$$\eta_2 = \left[\int_{x_4}^{x_3} (M_0/a_0^{n/2})\,dx/(1 - x_4 e_{max} - (1 - x_3)e_{min})\right]^{1/2} \equiv K(M_0, a_0). \tag{4.8}$$

The iteration scheme for Problem II takes the following form.

(i) Choose $a_0^{(0)}$ and $e_0^{(0)}$ as the solutions in the unconstrained case computed for Problem I, with n replaced by $n+1$.

(ii) In the constrained case let

$$e_0^{(1)} = e_{min} \quad \text{if} \quad e_0^{(0)} \leq e_{min}, \quad e_0^{(1)} = e_{max} \quad \text{if} \quad e_0^{(0)} \geq e_{max},$$
$$e_0^{(1)} = e_0^{(0)} \quad \text{otherwise}$$

(iii) Go to (4.7) with $e(x) = e_0^{(1)}(x)$ and impose thickness constraints on a_0 if there are any.

(iv) $\eta_2^{(i+1)} = K(M_0^{(i)}, a_0^{(i)})$. (4.9)

(v) Let $\bar{e}^{(i)} = M_0^{(i)}/(\eta_2^{(i)} a_0^{(i)n})^{1/2}$ and define

$$e_0^{(i+1)} = \begin{cases} e_{min} & \text{if} \quad \bar{e}^{(i)} \leq e_{min}, & x \in [x_3, 1] \\ \bar{e}^{(i)} & \text{if} \quad e_{min} < \bar{e}^{(i)} < e_{max}, & x \in [x_4, x_3] \\ e_{max} & \text{if} \quad \bar{e}^{(i)} \geq e_{max}, & x \in [0, x_4]. \end{cases}$$

We note that this step also determines x_3 and x_4.

(vi) Go to (4.7) with $e(x) = e_0^{(i+1)}(x)$.

(vii) Return to (4.9) if $e_0^{(i+1)}(x)$ and $\lambda_{ae}^{(i+1)}$ are non-stationary, else terminate.

The procedures starting at (4.7) and (4.9) constitute the double iteration scheme.

Figure 9 shows the curves of λ_{ae}/λ_a plotted vs e_{min} for $n = 1, 2, 3$ with $a_{min} = 0$, $\beta = p = 0$, $q_0 = 1$. The values of λ_a correspond to a homogeneous column ($e = 1$). We see that the efficiency of a design can be considerably increased by optimally designing the distribution of non-homogeneity in addition to the distribution of thickness.

Figure 10 gives the optimal distributions of shape and non-homogeneity in the unconstrained and constrained cases for $n = 2$, $\beta = p = 0$. We observe that imposing lower bounds on e_0 and a_0 increases $a_0(0)$ and $e_0(0)$, respectively.

The results of the paper can be stated in the form of isoperimetric inequalities. For Problem I, we make use of the values of k_n given in Section 2, (2.10), (2.14), (3.5), (3.9), (3.15) to obtain

$$\lambda_0 \leq \frac{C_n}{L^{n+1}}\left(\int_0^L A(X)\,dX\right)^n \left(\int_0^L E(X)\,dX\right)\left(\int_0^L Q(X)dX\right)^{-1} R(n) \tag{4.10}$$

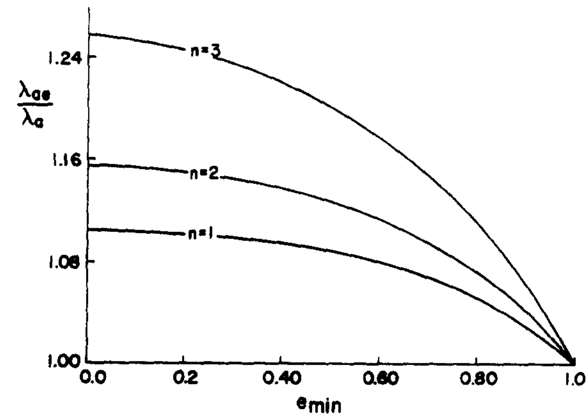

Fig. 9. Curves of λ_{ae}/λ_a plotted vs e_{min} with $a_{min} = 0$, $\beta = p = 0$ and $q_0 = 1$. Values of λ_a correspond to an optimal homogeneous column.

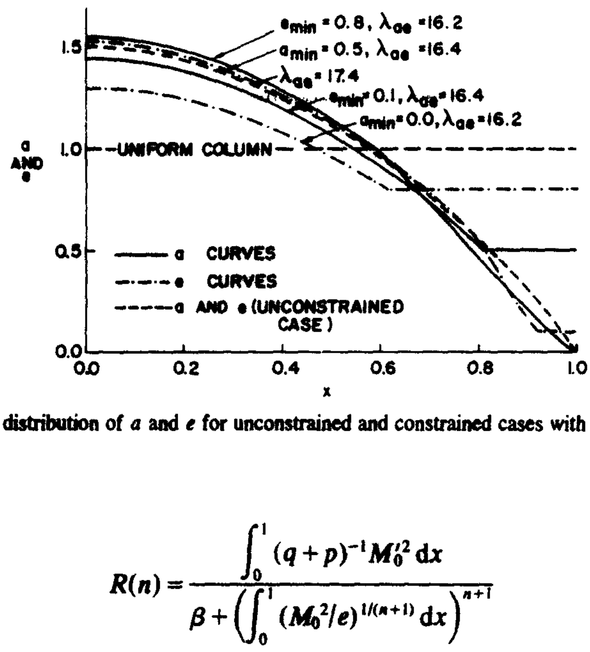

Fig. 10. Optimal distribution of a and e for unconstrained and constrained cases with $n = 2$, $\beta = p = 0$.

where

$$R(n) = \frac{\int_0^1 (q+p)^{-1} M_0'^2 \, dx}{\beta + \left(\int_0^1 (M_0^2/e)^{1/(n+1)} \, dx \right)^{n+1}} \tag{4.11}$$

with M_0 denoting the solution of (2.12), (2.13), (3.15) for $a = a_0$. Here C_n is a constant depending on n and the geometry of the cross section. From the values of k_n given in Section 2, we obtain $C_1 = H^2$, $C_2 = \bar{k}$, $C_3 = 1/12B^2$ for rectangular cross sections and $C_1 = D^2/8$, $C_3 = 1/8\pi^2 t^2$ for thin-walled cylinders. $R(n)$ is, in fact, equal to λ_a when $a_{min} = 0$, $a_{max} = \infty$. $R(n)$ can be obtained from Figs. 3, 6 and 8 for various values of the parameters n and p and the functions $q(x)$ and $e(x)$ with $\beta = 0$. The case $\beta > 0$ requires another figure similar to Fig. 5 but with ordinate λ_a instead of λ_a/λ_{ue}. In the special case $n = 1$, M_0 can be evaluated explicitly and $R(1)$ is given by

$$R(1) = \frac{\alpha_1 + \beta\alpha_2 - [(\alpha_1 - \beta\alpha_2)^2 + 4\beta\alpha_3]^{1/2}}{2\beta(\alpha_1\alpha_2 - \alpha_3^2)} \tag{4.12}$$

where

$$\alpha_1 = \int_0^1 (q+p)\left(\int_0^x \bar{e}^{1/2} \, d\xi\right)^2 dx, \quad \alpha_2 = \int_0^1 (q+p)\, dx, \quad \alpha_3 = \int_0^1 (q+p)\left(\int_0^x \bar{e}^{1/2} \, d\xi\right) dx. \tag{4.13}$$

We note that $\alpha_1\alpha_2 - \alpha_3^2 > 0$, owing to Hölder s inequality and the special form of α_1, α_2 and α_3 in (4.13). In the special case $\beta = 0$ we have $R(1) = \alpha_1^{-1}$.

For Problem II, the isoperimetric inequality is again given by (4.10) with $R(n)$ replaced by $R(n+1)$ and e set equal to 1 in (4.11).

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